# ME-221
# SOLUTIONS FOR PROBLEM SET 3

## Problem 1

a) To analyze the system, we begin with the electrical circuit. The back-emf (the voltage induced by the rotation of the motor) is given by $E_m(t) = K_m\omega(t)$.

$$u(t) = R_m i_m(t) + K_m\omega(t) \tag{1}$$

Next, we must write the equation of motion as torque balance on the motor shaft. There are three torques acting on the motor: (i) The torque driving the motor, which is provided by the electrical current $i_m(t)$ through the windings and given by $M_m = K_t i_m(t)$, (ii) the rotational dampening torque, given by $f_R\omega(t)$ and (iii) the torque exercised by the translational elements (mass $M$ and translational viscous damper $f_T$) via the pinion gear, which we'll denote by $T_p$. The resulting equation of motion is:

$$J\dot{\omega}(t) = K_t i_m(t) - f_R\omega(t) - T_p(t) \tag{2}$$

Next, we write down the equation of motion of the translational part of the load:

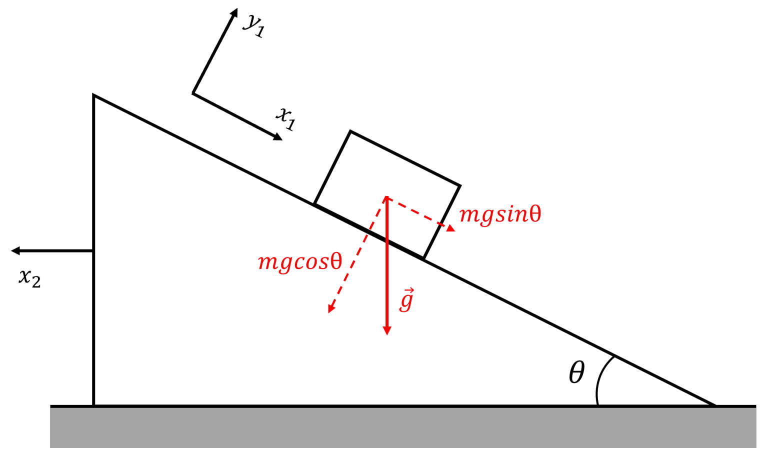
$$M\dot{v}(t) = F_p(t) - f_T v(t) \tag{3}$$

where, because of the transmission between the rack and pinion at $r$, we have: $F_p(t) = T_p(t)/r$ and $v(t) = r\omega(t)$.

b) The system is linear. We assumed that there is no friction between the wheels and the floor, the motion of the mass is only translational, the motor operates without losses, all elements are massless except the motor and the object attached to the pinion.

# Problem 2

a) The coordinate system of the block will be aligned with the inclined plane and that of the wedge with the expected direction of acceleration, as shown in the figure below:




The kinetic energies of the block and of the wedge can be written as:

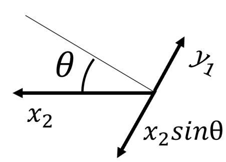
$$T_m = \frac{1}{2}m(\dot{x_1}^2 + \dot{y_1}^2) \tag{4}$$

$$T_M = \frac{1}{2}M\dot{x_2}^2 \tag{5}$$

And the potential energy:

$$V_m = -mgsin(\theta)x_1 + mgcos(\theta)y_1 \tag{6}$$

To satisfy the requirement that the block remains in contact with the inclined plane, we can impose that the displacement of the block in the direction perpendicular to that of the inclined plane must be equals and opposite to the displacement of the wedge in the same direction, as shown in the figure below.




Therefore we get:

$$y_1 = -x_2 sin(\theta) \tag{7}$$

By taking the first derivative, we get:

$$\dot{y_1} = -\dot{x_2} sin(\theta) \tag{8}$$

We can therefore rewrite equation 4 and 6 as:

$$T_m = \frac{1}{2}m(\dot{x_1}^2 + \dot{x_2}^2 sin^2(\theta)) \tag{9}$$
$$V_m = -mgsin(\theta)x_1 - mgsin(\theta)cos(\theta)x_2 \tag{10}$$

The Lagrangian is therefore:

$$L = T - V = T_m + T_M - V_m \tag{11}$$
$$L = \frac{1}{2}m(\dot{x_1}^2 + \dot{x_2}^2 sin^2(\theta)) + \frac{1}{2}M\dot{x_2}^2 + mgsin(\theta)x_1 + mgsin(\theta)cos(\theta)x_2 \tag{12}$$

We can therefore calculate the following:

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{x_1}}) = m\ddot{x_1} \tag{13}$$
$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{x_2}}) = (M + msin^2(\theta))\ddot{x_2} \tag{14}$$
$$\frac{\partial L}{\partial x_1} = mgsin(\theta) \tag{15}$$
$$\frac{\partial L}{\partial x_2} = mgsin(\theta)cos(\theta) \tag{16}$$

The resulting equations of motion (Euler-Lagrange) are:

$$m\ddot{x_1} = mgsin(\theta) \tag{17}$$
$$(M + msin^2(\theta))\ddot{x_2} = mgsin(\theta)cos(\theta) \tag{18}$$

b) Equation 18 can be rewritten to obtain the same result as with Newton's method:

$$\ddot{x_2} = \frac{mgsin(\theta)cos(\theta)}{M + msin^2(\theta)} \tag{19}$$

# Problem 3

We are given that $x$ is the coordinate of $M$, and $\theta$ is the angle of the pendulum. Then the position and the velocity of the mass $m$ in Cartesian coordinates are:

$$x_m = x + l\sin(\theta) \tag{20}$$
$$y_m = -l\cos(\theta) \tag{21}$$
$$\dot{x_m} = \dot{x} + l\dot{\theta}cos(\theta) \tag{22}$$
$$\dot{y_m} = l\dot{\theta}sin(\theta) \tag{23}$$

The Lagrangian is therefore:

$$L = T_M + T_m - V_m \tag{24}$$

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + l^2\dot{\theta}^2 + 2l\dot{x}\dot{\theta}\cos(\theta)) + mgl\cos(\theta) \tag{25}$$

We can therefore calculate the following:

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{x}}) = (M+m)\ddot{x} + ml\ddot{\theta}cos(\theta) - ml\dot{\theta}^2 sin(\theta) \tag{26}$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}}) = ml^2\ddot{\theta} + ml\ddot{x}cos\theta - ml\dot{x}\dot{\theta}sin(\theta) \tag{27}$$

$$\frac{\partial L}{\partial x} = 0 \tag{28}$$

$$\frac{\partial L}{\partial \theta} = -ml\dot{x}\dot{\theta}sin(\theta) - mglsin(\theta) \tag{29}$$

The resulting equations of motion (Euler-Lagrange) are:

$$(M+m)\ddot{x} + ml\ddot{\theta}\cos(\theta) - ml\dot{\theta}^2\sin(\theta) = 0$$
$$l\ddot{\theta} + \ddot{x}\cos(\theta) + g\sin(\theta) = 0$$

# Problem 4

The kinetic energy, potential energy and the Rayleigh's dissipation function of the system are given by:

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2$$
$$V = \frac{1}{2}k(x_1 - x_2)^2 + m_1gx_1 + m_2gx_2$$
$$D = \frac{1}{2}f(\dot{x}_1 - \dot{x}_2)^2$$

The equations of motion are:

$$m_1\ddot{x}_1 + f(\dot{x}_1 - \dot{x}_2) + k(x_1 - x_2) + m_1g = 0$$
$$m_2\ddot{x}_2 - f(\dot{x}_1 - \dot{x}_2) - k(x_1 - x_2) + m_2g = 0$$